THE DIELECTRIC ANALOGUE OF MAGNETOHYDRODYNAMICS*

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Abstract. Almost half a century after Alfvén first conceived of the science of magnetohydrodynamics, it is still possible to trace his intuitive thinking to explore physical processes heretofore not considered. The ideas of magnetohydrodynamics (applicable to conducting fluids) can be transferred almost intact to purely dielectric fluids, such that we can arrive at a generalized concept applicable to *any* fluid – conducting or dielectric. In this sense, Alfvén's conception of magnetohydrodynamics may be ideationally even more profound than it has been thought to be so far.

1. Historical Background

The essential ideas of electrodynamic interaction in matter had been formulated by the turn of the century by André Marie Ampère, James Clerk Maxwell, Hendrik Antoon Lorentz, and others. A significant ramification of these, however, was proposed by Hannes Alfvén in 1942 when he predicted a form of wave behaviour in a magnetized conducting fluid (Alfvén, 1942; Alfvén and Fälthammar, 1963) that later came to be known by his name. The principle underlying this phenomenon later formed the basis of multifarious developments in plasma physics and space physics. It was Alfvén's now legendary scientific intuition that led him to combine Maxwell's curl equations with Lorentz' force law ('a current-carrying conductor in a magnetic field experiences a force') and Newton's second law ('force equals mass times the acceleration') to deduce the science of magnetohydrodynamics – interactions in a magnetized *conducting* fluid involving the interchange of magnetic and kinetic energies. In the light of this same intuition, it is instructive to explore the Ampère–Maxwell–Lorentz–Alfvén connection to seek its lessons. In so doing, however, let us also invoke a simple aesthetic criterion: that of completeness.

A prefatory comment is appropriate at the outset of this discussion which is concerned with physical processes in an idealized situation. We discuss Alfvén's concept of magnetohydrodynamics only in that context – in its 'textbook' sense. It is now well-known that Alfvén has for some time sought to de-emphasize the frozen flow concept and taught against its indiscriminate application – especially in the case of tenuous plasmas in space (see, e.g., Alfvén, 1981). The present discussion of another type of frozen flow in another, very different, medium is not intended to detract from that effort and that teaching.

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2. The Magnetohydroelectric Effect

As is widely recognized now, the greatness of Maxwell's intuition was rooted in his conception of displacement current (free-space displacement current + polarization current in dielectrics) that completed the troubled Ampère's law (conceived with only conductors in mind), and provided an understanding of electromagnetic waves. Lorentz, in adding his force law to these results, again considered only conductors and did not say anything about dielectrics. And Alfvén then confined himself entirely to the conductors, leaving unaddressed the dielectrics. We thus begin to sense a certain incompleteness in the story of development of ideas, starting with Lorentz. If we postulate a 'Lorentz force' in a dielectric, we can restore completeness to this development.

The basic suggestion of the magnetic force on a pure dielectric material carrying a pure polarization current follows from simple, straightforward arguments. Let us recall first the derivation of the Lorentz force in the case of a conductor. A *conduction* current consists of a flow of electrons (each having a velocity $\mathbf{v}_i$ and a charge q, say) under the influence of an applied electric field $\mathbf{E}$. The total current density is $\mathbf{J} = \Sigma q\mathbf{v}_i$, where the summation is taken over all the electrons in a unit volume. Each moving electron experiences a force $\mathbf{f}_i = q\mathbf{v}_i \times \mathbf{B}$ in the presence of a magnetic field $\mathbf{B}$. These individual forces on all the electrons are transmitted to the body of the conductor through collisions with atoms. Thus the total force on a unit volume is $\mathbf{F} = \Sigma q\mathbf{v}_i \times \mathbf{B} = (\Sigma q\mathbf{v}_i) \times \mathbf{B} = \mathbf{J} \times \mathbf{B}$.

No such flow of charges occurs in a dielectric carrying a pure *polarization* current, and one is thus apt not to think in terms of the presence of a similar force. We note, however, that there is here nevertheless a microscopic displacement of the positive and the negative charges bound in the atoms and molecules of the dielectric, and these individual charges are subject to the same Lorentz force as the free electrons in a conductor; the positive charges q^+ and the negative charges q^- move in opposite directions under the influence of a time-varying field $\mathbf{E}$, with velocities $\mathbf{v}_i^+$ and $\mathbf{v}_i^-$, respectively. Thus the net *polarization* current is $\mathbf{J} = \Sigma(q^+\mathbf{v}_i^+ + q^-\mathbf{v}_i^-)$. The force on an individual positive charge is $\mathbf{f}_i^+ = q^+\mathbf{v}_i^+ \times \mathbf{B}$ and on a negative charge, $\mathbf{f}_i^- = q^-\mathbf{v}_i^- \times \mathbf{B}$. Clearly, they both point in the same direction, giving a net force on each individual atom or molecule. The total volume force is again formally given by $\mathbf{F} = \Sigma(\mathbf{f}_i^+ + \mathbf{f}_i^-) = \Sigma(q^+\mathbf{v}_i^+ + q^-\mathbf{v}_i^-) \times \mathbf{B} = \mathbf{J} \times \mathbf{B}$. From these simple arguments, we are now able to make an important generalization: *All substances (conductors and dielectrics) experience the* $\mathbf{J} \times \mathbf{B}$ *force in a magnetic field.* This provides the full complement of the force law to the Maxwell's equations.

We will now take the next logical step, and attempt to provide the dielectric counterpart of Alfvén's ideas.

Consider for simplicity a pure dielectric fluid (nonconducting, lossless) with a polarizability χ and a dielectric constant $\varepsilon = (1 + \chi)\varepsilon_0$ placed in a magnetic field $\mathbf{B}$, and moving with a velocity $\mathbf{u}$ (assumed nonrelativistic). Then the electric field $\mathbf{E}'$ in the body of the moving fluid is related to the field $\mathbf{E}$ in the laboratory (or rest) frame by (see, e.g., Stratton, 1941; Section 1.23)

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}, \tag{1}$$

so that the polarization current density in the body of the moving fluid is

$$\mathbf{J}' = \chi\varepsilon_0\dot{\mathbf{E}}'. \tag{2}$$

The current density in the rest frame may now be written as (*op. cit.*)

$$\mathbf{J} = \mathbf{J}' + \rho_p\mathbf{u}, \tag{3}$$

where ρ_p is the polarization space charge density in the fluid. This is the density of the induced polarization charges that arise in the body of the moving fluid, any free charges arising on a rigid bounding conductor. We now have

$$\mathbf{J} = \chi\varepsilon_0\left[\dot{\mathbf{E}} + \frac{\partial}{\partial t}(\mathbf{u}\times\mathbf{B})\right] + \rho_p\mathbf{u}. \tag{4}$$

This may be recognized as the dielectric equivalent of the 'Generalized Ohm's Law' of magnetohydrodynamics.

The volume force on the fluid consists of the electromagnetic $\mathbf{J}\times\mathbf{B}$ force, and the electrostatic force $\rho_p\mathbf{E}$ (cf. Stratton, 1941; Section 2.21; we assume here that the fluid is incompressible). Hence, the force balance equation (Newton's second law) for the fluid is

$$\frac{\partial}{\partial t}(\rho\mathbf{u}) = \mathbf{J}\times\mathbf{B} + \rho_p\mathbf{E}, \tag{5}$$

where ρ is the mass density of the fluid. Other force terms due to gravity, pressure gradient, etc., are possible. The above relation is the fundamental force equation of magnetohydrodynamic interaction *in a dielectric fluid.* It involves an interchange of magnetic, fluid-kinetic *and* electrostatic energies. For this reason, the term 'magnetohydroelectric interaction' was proposed to describe the phenomenon (De, 1979a,b, 1980).

3. The Equation of Magnetohydroelectrics: Magnetic Field Freezing and Dielectric 'Alfvén Waves'

In order to develop our discussion further, it is necessary to obtain a relationship between the bound polarization charge density ρ_p and the electric field $\mathbf{E}$. Consideration of the charge build-up on an elementary capacitor will show that

$$\nabla\cdot\chi\varepsilon_0\mathbf{E} = -\rho_p. \tag{6}$$

Recall now the Maxwell's equations

$$\nabla\cdot\mathbf{B} = 0, \tag{7}$$

$$\nabla\times\mathbf{B} = \mu\mathbf{J} + \mu\varepsilon_0\dot{\mathbf{E}}, \tag{8}$$

$$\nabla\times\mathbf{E} = -\dot{\mathbf{B}}; \tag{9}$$

where μ is the magnetic permeability of the fluid. Using Equations (7), (8), and (9) in Equation (4), we are now able to derive

$$\ddot{\mathbf{B}} = \frac{1}{\mu\varepsilon}\nabla \cdot \mathbf{B} + \frac{\chi}{1+\chi}\frac{\partial}{\partial t}\nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{1}{\varepsilon}\nabla \times (\rho_p \mathbf{u}). \quad (10)$$

This equations contains all the essence of the magnetohydroelectric interaction when taken in conjunction with Equation (5). While this latter equation does not enter directly into the derivation of Equation (10), it determines whether or not the magnetohydroelectric effect is significant (Equation (10) by itself could be satisfied for arbitrarily small values of **B**; see De, 1979b).

When the first term on the right-hand side of Equation (10) dominates, we have

$$\ddot{\mathbf{B}} = \frac{1}{\mu\varepsilon}\nabla^2 \mathbf{B}, \quad (11)$$

the familiar equation for three-dimensional electromagnetic wave propagation in a dielectric. When the second term dominates and $\chi \gg 1$, we arrive at the well-known condition for frozen flow (cf. Alfvén and Fälthammar, 1963)

$$\ddot{\mathbf{B}} = \frac{\partial}{\partial t}\nabla \times (\mathbf{u} \times \mathbf{B}). \quad (12)$$

Thus, in a medium radically different from what Alfvén was concerned with, we have arrived at the same physical condition he had envisioned. In this state of frozen flow, it is also possible to deduce an Alfvén wave-like wave behaviour (the 'magnetohydroelectric wave'; see De, 1979a). Such waves propagate along magnetic field lines with a velocity v given by

$$v^2 = c^2\left(\frac{1}{\chi} + \frac{v_A^2}{c^2}\right)\left(1 + \frac{1}{\chi} + \frac{v_A^2}{c^2}\right)^{-1}, \quad (13)$$

where c is the velocity of light in free space, and $v_A = B/(\mu\rho)^{1/2}$ is again a familiar parameter that makes its appearance: the Alfvén velocity. It has further been shown (*op. cit.*) that a fully generalized wave behaviour can be derived in an arbitrary medium which is partly conducting and partly dielectric, and that in various appropriate limits this wave reduces to the ordinary electromagnetic wave, the Alfvén wave and the magnetohydroelectric wave.

4. Magnetic Flux Amplification and Electric Field Freezing

Our discussion so far has developed in close parallelism with conventional magnetohydrodynamics. We now wish to venture somewhat far afield to explore if anything more can be gleaned from our formulation thus far. When the last term on the right-hand side

Equation (10) dominates, there arises a state described by

$$\ddot{\mathbf{B}} = \frac{1}{\varepsilon} \nabla \times (\rho_p \mathbf{u}). \tag{14}$$

The induced magnetic field may now be perpendicular to the fluid motion, and parallel to the original static magnetic field. This is *not* the state of frozen-in magnetic field lines; rather, it indicates an amplification of the magnetic flux resulting from an exchange of energy among the three fields (magnetic, electric, and velocity). A magnetic flux tube here may be imagined to be constricted. As we shall see below, this is in fact a state of frozen-in electric field lines (a concept that would be meaningless in the case of a perfectly conducting fluid).

Upon using Equations (6) and (9) in the above equation, we obtain

$$\dot{\mathbf{E}} = -(\nabla \cdot \mathbf{E})\mathbf{u}, \tag{15}$$

$$\dot{\mathbf{E}} = \frac{1}{\varepsilon} \rho_p \mathbf{u}; \tag{16}$$

the implications of which are immediately obvious: the change in the electric field **E** equals the divergence of the field times the fluid velocity – i.e., it owes itself to a movement of the polarization space charges along with the fluid. The electric field lines are 'tied' to these space charges and move along with them.

5. Remarks

From the above discussion, it follows simply that one could erect a generalized formalism for magnethydrodynamic interaction in a fluid of generalized property (conducting *and* dielectric). In the limit of infinite conductivity, such a formalism would lead to the state of frozen-in magnetic field lines; in the other limit, that of infinitely high dielectric constant, a state of frozen-in electric field lines is possible. In this sense, Alfvén's conception of magnetohydrodynamics may be ideationally even more profound in its scope than it has been thought to be so far. The physical realms of manifestation of the conducting and the dielectric effects, however, differ greatly. Frozen flow in conducting fluids is favoured for low frequencies and large length scales; in dielectric fluids quite the opposite is true. Thus the realms of applicability are also very different. Whereas the former effect has found application in space science and in large scale devices in the industry, the latter effect – whose applications, if any, may well lie far into the future – will conceivably apply to experiments and devices involving motions at microscopically small physical length scales.

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